NONSIMPLE POLYOMINOES AND PRIME IDEALS

TAKAYUKI HIBI AND AYESHA ASLOOB QURESHI

Abstract. It is known that the polyomino ideal arising from a simple polyomino comes from a finite bipartite graph and, in particular, it is a prime ideal. A class of nonsimple polyominoes $\mathcal{P}$ for which the polyomino ideal $I_{\mathcal{P}}$ is a prime ideal and for which $I_{\mathcal{P}}$ cannot come from a finite simple graph will be presented.

Introduction

The systematic study of the binomial ideals arising from polyominoes originated in the work [9] by the second author. First, we briefly recall fundamental materials and basic terminologies on polyominoes and their binomial ideals. We refer the reader to [9] for further information on algebra and combinatorics on polyominoes.

(0.1) Let $\mathbb{N}$ denote the set of nonnegative integers and

$$\mathbb{N}^2 = \{(i, j) \,:\, i, j \in \mathbb{N}\}.$$

Given $a = (i, j)$ and $b = (k, \ell)$ belonging to $\mathbb{N}^2$, we write $a < b$ if $i < k$ and $j < \ell$.

When $a < b$, we define an *interval* $[a, b]$ of $\mathbb{N}^2$ to be

$$[a, b] = \{c \in \mathbb{N}^2 \,:\, a \leq c \leq b\} \subset \mathbb{N}^2.$$

For an interval $[a, b]$, the *diagonal* corners of $[a, b]$ are a and b, and the *anti-diagonal* corners of $[a, b]$ are $c = (i, \ell)$ and $d = (k, j)$.

(0.2) A *cell* of $\mathbb{N}^2$ with the lower left corner $a \in \mathbb{N}^2$ is the interval $C = [a, a+(1, 1)]$. Its *vertices* are $a, a + (1, 0), a + (0, 1)$ and $a + (1, 1)$. Its *edges* are

$$\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}, \{a + (0, 1), a + (1, 1)\}.$$

Let $V(C)$ denote the set of vertices of C and $E(C)$ the set of edges of C.

(0.3) Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. Then its *vertex set* is $V(\mathcal{P}) = \bigcup_{C\in\mathcal{P}} V(C)$ and its *edge set* is $E(\mathcal{P}) = \bigcup_{C\in\mathcal{P}} E(C)$. Let C and D be cells of $\mathcal{P}$. We say that C and D are *connected* if there exists a sequence of cells

$$\mathcal{C} \,:\, C = C_1, \ldots, C_m = D$$

of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m - 1$. Furthermore, if $C_i \neq C_j$ for all $i \neq j$, then $\mathcal{C}$ is called a *path* connecting C with D.

We say that $\mathcal{P}$ is a *polyomino* if any two cells of $\mathcal{P}$ are connected. A polyomino Q is a *subpolyomino* of $\mathcal{P}$ if each cell belonging to $\mathcal{Q}$ belongs to $\mathcal{P}$.

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(0.4) Let A and B be cells of $\mathbb{N}^2$ for which (i, j) is the lower left corner of A and (k, ℓ) is the lower left corner of B. If $i \leq k$ and $j \leq \ell$, then the *cell interval* of A and B is the set $[A, B]$ which consists of those cells E of $\mathbb{N}^2$ whose lower left corner (r, s) satisfies $i \leq r \leq k$ and $j \leq s \leq \ell$.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. We call $\mathcal{P}$ *row convex* if the horizontal cell interval $[A, B]$ is contained in $\mathcal{P}$ for any cells A and B of $\mathcal{P}$ whose lower left corners are in horizontal position. Similarly one can define *column convex*. We call $\mathcal{P}$ *convex* if it is row convex and column convex.

An edge of $\mathcal{P}$ is a *free* edge if it is an edge of only one cell of $\mathcal{P}$. The *boundary* $B(\mathcal{P})$ of $\mathcal{P}$ is the union of all free edges of $\mathcal{P}$. A cell C of $\mathcal{P}$ is a *border* cell if at least one of the edges of C is a free edge.

(0.5) Each interval $[a, b]$ of $\mathbb{N}^2$ can be regarded as a polyomino in the obvious way. This polyomino is denoted by $\mathcal{P}_{[a,b]}$. Let $\mathcal{P}$ be a collection of cells of $\mathbb{N}^2$ and $[a, b] \subset \mathbb{N}^2$ an interval with $\mathcal{P} \subset \mathcal{P}_{[a,b]}$. Following [9], we say that a polyomino $\mathcal{P}$ is *simple* if, for any cell C of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, C_2, \ldots, C_m = D$ with each $C_i \notin \mathcal{P}$ such that D is not a cell of $\mathcal{P}_{[a,b]}$. Roughly speaking, a simple polyomino is a polyomino with no "hole" (see [9, Figure 3]).

(0.6) Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$ with $V(\mathcal{P})$ its vertex set. Let S denote the polynomial ring over a field K whose variables are those x_a with $a \in V(\mathcal{P})$. We say that an interval $[a, b]$ of $\mathbb{N}^2$ is *an interval of* $\mathcal{P}$ if $\mathcal{P}_{[a,b]} \subset \mathcal{P}$. For each interval $[a, b]$ of $\mathcal{P}$, we introduce the binomial

$$f_{a,b} = x_a x_b - x_c x_d,$$

where c and d are the anti-diagonals of $[a, b]$. Such a binomial $f_{a,b}$ is said to be an *inner* 2*-minor* of $\mathcal{P}$. Write $I_{\mathcal{P}}$ for the ideal generated by all inner 2-minors of $\mathcal{P}$. Especially, when $\mathcal{P}$ is a polyomino, we say that $I_{\mathcal{P}}$ is the *polyomino ideal* of $\mathcal{P}$.

Now, one of the most exciting algebraic problems on polyominoes is when a polyomino ideal is a prime ideal. It is known ([4] and [8]) that if a polyomino $\mathcal{P}$ is simple, then its polyomino ideal $I_{\mathcal{P}}$ is a prime ideal. The polyomino ideals arising from simple polyominoes, however, turn out to be well-known ideals [7] arising from Koszul bipartite graphs. Thus, form a view point of finding a new class of binomial prime ideals, it is reasonable to study polyomino ideals of nonsimple polyominoes. In the present paper, a class of nonsimple polyominoes $\mathcal{P}$ for which the polyomino ideal $I_{\mathcal{P}}$ is a prime ideal (Theorem 2.1) and for which $I_{\mathcal{P}}$ cannot come from a finite simple graph (Theorem 3.1) will be presented.

Finally the fact [1] that a binomial ideal is a prime ideal if and only if it is a toric ideal ([5, Chapter 5]) explains the reason why we are interested in polyomino ideals which are prime.

1. Gröbner bases of polyomino ideals

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. Let, as before, S denote the polynomial ring over a field K whose variables are those x_a with $a \in V(\mathcal{P})$. We work with the lexicographical order on S induced by the ordering of the variables x_a, $a \in V(\mathcal{P})$, such that $x_a > x_b$ with $a = (i, j)$ and $b = (k, \ell)$, if $i > k$, or, $i = k$ and $j > \ell$.

We refer the reader to [2, Chapter 2] and [5, Chapter 1] for basic terminologies and results on Gröbner bases.

Lemma 1.1 ([9])**.** *Let $\mathcal{P}$ be a collection of cells of $\mathbb{N}^2$. Then the set of inner 2-minors of $\mathcal{P}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<_{\text{lex}}$ if and only if, for any two intervals $[a, b]$ and $[b, c]$ of $\mathcal{P}$, either $[e, c]$ or $[d, c]$ is an interval of $\mathcal{P}$, where d and e are the anti-diagonal corners of $[a, b]$.*

Corollary 1.2. *Let $\mathcal{I} \subset \mathbb{N}^2$ be an interval of $\mathbb{N}^2$ and $\mathcal{P}$ a convex polyomino which is a subpolyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P}^c = \mathcal{P}_{\mathcal{I}} \setminus \mathcal{P}$. Then the set of inner 2-minors of $\mathcal{P}^c$ forms a reduced Gröbner basis of $I_{\mathcal{P}^c}$ with respect to $<_{\text{lex}}$.*

Proof. Suppose that there exist intervals $[a, b]$ and $[b, c]$ of $\mathcal{P}^c$ such that neither $[e, c]$ nor $[d, c]$ is an interval of $\mathcal{P}^c$, where d and e are the anti-diagonal corners of $[a, b]$. Then one can choose a cell C of $\mathcal{P}_{[e,c]}$ and a cell D of $\mathcal{P}_{[d,c]}$ such that C and D belong to $\mathcal{P}$. Now, since $\mathcal{P}$ is a polyomino, it follows that there is a path of cells $C = C_1, C_2, \ldots, C_n = D$ of $\mathcal{P}$ connecting C with D. Then one of the situations drawn in Figure 1 occurs. Let $C = [a', a' + (1, 1)]$.

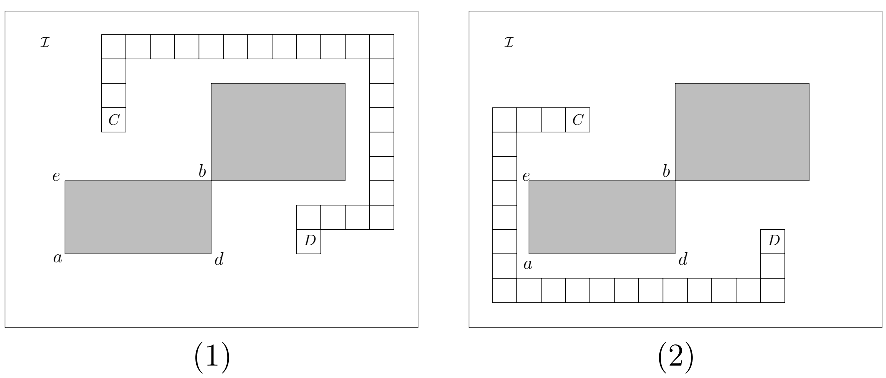

FIGURE 1.

In other words, there is $1 < j < n$ for which $C_j = [c', c' + (1, 1)]$ satisfies one of the followings:

(i) if $a' = (\xi', \nu'), c' = (\xi'', \nu''), c = (\xi, \nu)$, then $\nu' = \nu''$ and $\xi'' > \xi$;

(ii) if $a' = (\xi', \nu'), c' = (\xi'', \nu''), a = (\xi_0, \nu_0)$, then $\xi' = \xi''$ and $\nu'' < \nu_0$.

Since $\mathcal{P}$ is convex, it follows that, in (i) one has $[C, C_j] \subset \mathcal{P}$, and in (ii) one has $[C_j, C] \subset \mathcal{P}$. However, $[C, C_j] \cap \mathcal{P}_{[a,b]} \neq \emptyset$ in (i) and $[C_j, C] \cap \mathcal{P}_{[a,b]} \neq \emptyset$ in (ii), each of which contradicts $\mathcal{P} \cap \mathcal{P}^c = \emptyset$. $\square$

2. Nonsimple polyominoes whose polyomino ideals are prime

We now come to the main result of the present paper.

Theorem 2.1. *Let $\mathcal{I} \subset \mathbb{N}^2$ be an interval of $\mathbb{N}^2$ and $\mathcal{P}$ a convex polyomino which is a subpolyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P}^c = \mathcal{P}_{\mathcal{I}} \setminus \mathcal{P}$ and suppose that $\mathcal{P}^c$ is a polyomino. Then the polyomino ideal $I_{\mathcal{P}^c}$ is a prime ideal.*

Proof. We may assume that $B(\mathcal{P}) \cap B(\mathcal{P}_{\mathcal{I}}) = \emptyset$; otherwise, $\mathcal{P}$ is a simple polyomino (see Figure 2) and, as was stated, the result follows from [4] and [8].

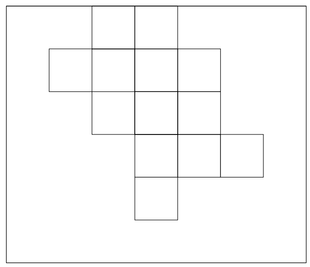

FIGURE 2.

Let $\mathcal{I} = [a, b]$ and c and d be the anti-diagonal corners of $[a, b]$, where b and c are in horizontal position. It follows from Theorem 1.2 that x_c cannot divide the initial monomial of any binomial belonging to the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<_{\text{lex}}$. Hence x_c is a nonzero divisor of $S/\text{in}_{<_{\text{lex}}}(I_{\mathcal{P}^c})$ and thus x_c is a nonzero divisor of $S/I_{\mathcal{P}^c}$ as well. Hence the localization map $S/I_{\mathcal{P}^c} \to (S/I_{\mathcal{P}^c})_{x_c}$ is injective. Here $(S/I_{\mathcal{P}^c})_{x_c}$ is the localization of $(S/I_{\mathcal{P}^c})_{x_c}$ at x_c. Thus, in order to prove that $S/I_{\mathcal{P}^c}$ is an integral domain, it suffices to show that $(S/I_{\mathcal{P}^c})_{x_c} = S_{x_c}/(I_{\mathcal{P}^c})_{x_c}$ is an integral domain. For this, we will show that $(I_{\mathcal{P}^c})_{x_c} = I_{\mathcal{P}'}$, where $\mathcal{P}'$ is a simple subpolyomino of $\mathcal{P}^c$, which guarantees that $(I_{\mathcal{P}^c})_{x_c}$ is a prime ideal ([4] and [8]).

Let $\mathcal{A} = \{p_1, \ldots, p_n\}$ denote the set of those $p_i \in V(\mathcal{P}^c)$ for which there is an interval $[r_i, q_i]$ of $\mathcal{P}^c$ whose anti-diagonal corners are c and p_i. See Figure 3.

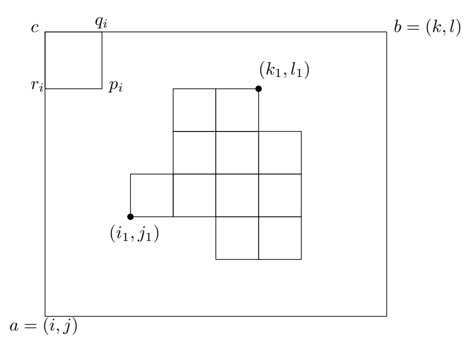

FIGURE 3.

One has $r_i \in [a, c]$ and $q_i \in [c, b]$. Since $x_{r_i}x_{q_i} - x_c x_{p_i} \in I_{\mathcal{P}^c}$ and since the variable x_c is invertible in S_{x_c}, one has $x_{p_i} = x_{q_i}x_{r_i}x_c^{-1}$ in $S_{x_c}/(I_{\mathcal{P}^c})_{x_c}$ Thus, in $S_{x_c}/(I_{\mathcal{P}^c})_{x_c}$, the variables x_{p_i} with $p_i \in \mathcal{A}$ can be ignored.

Let p_i and p_j belong to $\mathcal{A}$ for which $[p_i, p_j]$ is an interval in $\mathcal{P}^c$. It then follows that the anti-diagonals of $[p_i, p_j]$ are also contained in $\mathcal{A}$. Thus $f_{p_i,p_j} = x_{p_k}x_{p_\ell} - x_{p_j}x_{p_i}$, where p_k and p_ℓ are the anti-diagonal corners of $[p_i, p_j]$.

Let $[v, p_i]$ be an interval of $\mathcal{P}^c$ with $p_i \in A$ and $v \notin A$, then by using the fact that $[r_i, p_i] \setminus \{r_i\} \subset A$, it follows that the anti-diagonal corner $p_{i'}$ of $[v, p_i]$ which is in horizontal position with p_i belongs to A. Let v' be the other anti-diagonal corner of $[v, p_i]$. Since $r_i = r_{i'}$, the inner 2-minor $x_v x_{p_i} - x_{v'} x_{p_{i'}} \in I_{\mathcal{P}^c}$ can be written as $x_{r_i}(x_v x_{q_i} - x_{v'} x_{q_{i'}})$ in $(I_{\mathcal{P}^c})_{x_c}$. Hence $x_v x_{p_i} - x_{v'} x_{p_{i'}}$ is a multiple of $x_v x_{q_i} - x_{v'} x_{q_{i'}}$ in $(I_{\mathcal{P}^c})_{x_c}$. Similarly, if $[p_i, v]$ is an interval of $\mathcal{P}^c$ with $p_i \in A$ and $v \notin A$ and if $p_{i'} \in A$ and $v' \notin A$ are the anti-diagonal corner of $[p_i, v]$, then $x_v x_{p_i} - x_{v'} x_{p_{i'}}$ is a multiple of $x_v x_{r_i} - x_{v'} x_{r_{i'}}$ in $(I_{\mathcal{P}^c})_{x_c}$.

Let $\mathcal{P}'$ be the collection of cells contained in $\mathcal{P}^c$ obtained by removing all the cells that appear in $\bigcup_{i=1}^{n} \mathcal{P}_{[r_i, q_i]}$. Let $a = (i, j)$ and $b = (k, \ell)$. Then $c = (i, \ell)$. We choose $(i_1, j_1) \in V(\mathcal{P})$ such that, for any $(i_2, j_2) \in V(\mathcal{P})$, one has either $i_1 < i_2$ or $(i_1 = i_2$ and $j_1 < j_2)$. Similarly, we choose $(k_1, \ell_1) \in V(\mathcal{P})$ such that, for any $(k_2, \ell_2) \in V(\mathcal{P})$, one has either $\ell_1 > \ell_2$ or $(\ell_1 = \ell_2$ and $k_1 > k_2)$. In $V(\mathcal{P}')$, we identify the vertical interval $[a, (i, j_1)]$ with $[(i_1, j), (i_1, j_1)]$, and the horizontal interval $[(k_1, \ell_1), (k, \ell_1)]$ with $[(k_1, \ell), b]$. Then, with this identification and by using the above discussion, one has $I_{\mathcal{P}'} = (I_{\mathcal{P}^c})_{x_c}$.

Now, what we must prove is that $\mathcal{P}'$ is a simple polyomino. First we claim that $\mathcal{P}'$ is a polyomino. Let $\mathcal{B}$ be the collection of border cells of $\mathcal{P}_{[a,b]}$ belonging to $\mathcal{P}'$. Then $\mathcal{B}$ is connected. Since every cell of $\mathcal{P}'$ is connected to at least one of the cells belonging to $\mathcal{B}$. Hence $\mathcal{P}'$ is connected. Thus $\mathcal{P}'$ is a polyomino, as desired. Second, we claim that $\mathcal{P}'$ is simple. Let $\mathcal{J}$ be an interval such that $\mathcal{P}' \subset \mathcal{P}_{[a,b]} \subset \mathcal{P}_{\mathcal{J}}$. If $\mathcal{P}'$ is not a simple polyomino, then one has a cell $D \notin \mathcal{P}'$ for which every path connecting D with a cell not belonging to $\mathcal{P}_{\mathcal{J}}$ is interrupted by some cell of $\mathcal{P}'$. The inclusion $\mathcal{P}' \subset \mathcal{P}^c$ shows that D must be a cell of the convex polyomino $\mathcal{P}$. Then all the cells of $\mathcal{P}^c$ whose edge sets intersect $B(\mathcal{P})$ must be contained in $\mathcal{P}'$, which cannot be possible by our construction of $\mathcal{P}'$. Hence $\mathcal{P}'$ is simple, as required. $\square$

3. TORIC IDEALS OF FINITE GRAPHS

As was stated in Introduction, one of the most exciting algebraic problems on polyominoes is when a polyomino ideal is a prime ideal. The fact ([4] and [8]) that the polyomino ideals of simple polyominoes are prime seems to be of interest. However, it turns out that these binomial ideals belong to a subclass of binomial ideals arising from Koszul bipartite graphs ([7]). Thus, form a view point of finding a new class of binomial prime ideals, the study of polyomino ideals of nonsimple polyominoes is indispensable.

In fact, the polyomino ideals of Theorem 2.1 *cannot* come from finite simple graphs. (We say that a binomial ideal I comes from a finite simple graph if I coincides with a toric ideal [6] arising from a finite simple graph.) More generally, we can show that

Theorem 3.1. *Let $\mathcal{I} \subset \mathbb{N}^2$ be an interval of $\mathbb{N}^2$ and $\mathcal{P}$ a simple polyomino which is a subpolyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P}^c = \mathcal{P}_{\mathcal{I}} \setminus \mathcal{P}$ and suppose that $\mathcal{P}^c$ is a polyomino. Then its polyomino ideal cannot come from a finite simple graph.*

Proof. Let $\mathcal{J}$ be the smallest interval in $\mathbb{N}^2$ such that $\mathcal{P} \subset \mathcal{J}$. We choose $x_1, \dots, x_{16}$ belonging to $V(\mathcal{P}^c)$, as shown in Figure 4, where $\mathcal{P}$ is shown by grey region and where $\mathcal{J} = [x_{10}, x_7]$.

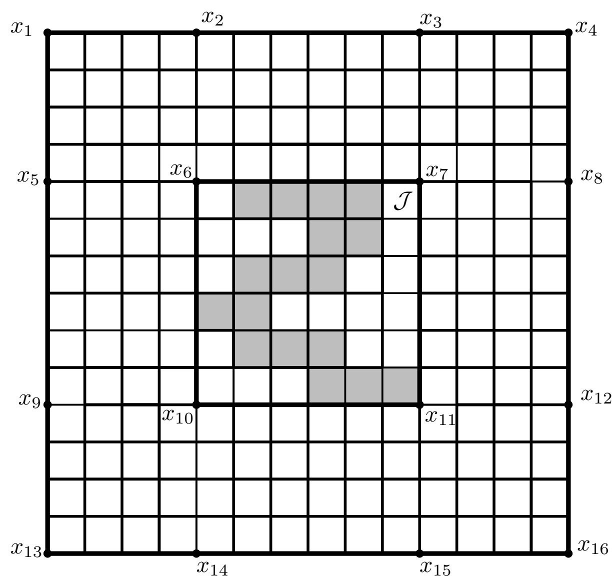

FIGURE 4. Polyomino $\mathcal{P}^c$

Assume that there exists a finite simple graph G with vertex set $V(G)$ and edge set $E(G)$ such that the toric ideal I_G arising from G is equal to $I_{\mathcal{P}}$. Let $K[G] = K[t_i t_j | \{i, j\} \in E(G)]$ be the edge ring of G. Then there exists an isomorphism $\phi : K[\mathcal{P}] \to K[G]$ such that for each $x_a \in K[\mathcal{P}]$ there exists a unique edge $\{i, j\} \in E(G)$ with $\phi(x_a) = t_i t_j$.

The 2-minor $x_2x_7 - x_3x_6$ is an inner minor of $\mathcal{P}^c$ and hence $\phi(x_2x_7) = \phi(x_3x_6)$. Let $\phi(x_2) = t_i t_j$. Then $\phi(x_7) = t_k t_l$ where i, j, k, l are pairwise distinct vertices of G and $\{i, j\}, \{k, l\} \in E(G)$. Then $\phi(x_3x_6) = t_i t_j t_k t_l$ which shows that we have one of the following possibilities:

(i) $\phi(x_3) = t_i t_k$ and $\phi(x_6) = t_j t_l$;
(ii) $\phi(x_3) = t_i t_l$ and $\phi(x_6) = t_j t_k$;
(iii) $\phi(x_3) = t_j t_k$ and $\phi(x_6) = t_i t_l$;
(iv) $\phi(x_3) = t_j t_l$ and $\phi(x_6) = t_i t_k$.

We may assume that $\phi(x_3) = t_i t_k$ and $\phi(x_6) = t_j t_l$. The discussion for other cases is similar. By using the inclusion $x_1x_6 - x_2x_5 \in I_{\mathcal{P}^c}$ and that $\phi(x_2) = t_i t_j$ and $\phi(x_6) = t_j t_l$, we see that $\phi(x_1) = t_i t_p$ and $\phi(x_5) = t_l t_p$where $\{i, p\}, \{l, p\} \in E(G)$ for some $p \in V(G) \setminus \{i, j, k, l\}$. Note that $p \neq k$ because otherwise $\phi(x_5) = \phi(x_7) = t_k t_l$, which is not possible. Now from $x_5x_{10} - x_6x_9 \in I_{\mathcal{P}^c}$ and $\phi(x_5) = t_p t_l$, $\phi(x_6) = t_j t_l$, we obtain $\phi(x_{10}) = t_j t_q$ and $\phi(x_9) = t_p t_q$ for some $q \in V(\mathcal{P}^c) \setminus \{i, p, l, j\}$. Continuing in the same way, from $x_9x_{14} - x_{10}x_{13} \in I_{\mathcal{P}^c}$ and $\phi(x_9) = t_p t_q$ and $\phi(x_{10}) = t_j t_q$, we get $\phi(x_{14}) = t_r t_j$ and $\phi(x_{13}) = t_r t_p$ for some $r \in V(\mathcal{P}^c) \setminus \{i, j, l, p, q\}$. Then, by using $x_{10}x_{15} - x_{11}x_{14} \in I_{\mathcal{P}^c}$, $\phi(x_{10}) = t_j t_q$ and $\phi(x_{14}) = t_r t_j$, we get $\phi(x_{15}) = t_s t_r$ and $\phi(x_{11}) = t_s t_q$ for some $s \in V(\mathcal{P}^c) \setminus \{j, p, q, r\}$.

Furthermore, by using $x_3x_8 - x_4x_7 \in I_{\mathcal{P}^c}$, $\phi(x_3) = t_it_k$ and $\phi(x_7) = t_kt_l$, we obtain $\phi(x_4) = t_it_y$ and $\phi(x_8) = t_lt_y$ for some $y \in V(G) \setminus \{i, k, l, j, p\}$. Similarly, from $x_7x_{12} - x_{11}x_8 \in I_{\mathcal{P}^c}$, $\phi(x_7) = t_kt_l$, $\phi(x_8) = t_yt_l$ and $\phi(x_{11}) = t_st_q$, it follows that $t_k|t_st_q$. Thus one has either $k = s$ and $\phi(x_{12}) = t_qt_y$ or $k = q$ and $\phi(x_{12}) = t_st_y$.

Let $k = s$. Then $\phi(x_6x_{11}-x_7x_{10}) = (t_jt_l)(t_kt_q)-(t_kt_l)(t_jt_q) = 0$, which guarantees $x_6x_{11}-x_7x_{10} \in I_G$. However, one has $x_6x_{11}-x_7x_{10} \notin I_{\mathcal{P}^c}$, because it is not an inner minor of $\mathcal{P}^c$, and it gives us a contradiction to our assumption $I_G = I_{\mathcal{P}^c}$. Hence $k = q$ and $\phi(x_{12}) = t_st_y$. But then $x_{11}x_{16} - x_{12}x_{15} \in I_{\mathcal{P}^c} = I_G$, $\phi(x_{12}x_{15}) = (t_st_y)(t_st_r)$ and $\phi(x_{11}) = t_st_k$. Thus one has either $k = r$ or $k = s$, which is not possible; otherwise either $\phi(x_{11}) = t_st_r = \phi(x_{15})$ or $\phi(x_{11}) = t_s^2$. As a result, we conclude that $I_G \neq I_{\mathcal{P}^c}$ for any finite simple graph G. $\square$

Finally, it may be conjectured that the polyomino ideal $I_{\mathcal{P}}$ of a polyomino $\mathcal{P}$ comes from a finite simple graph if and only if $\mathcal{P}$ is nonsimple. Furthermore, Theorem 2.1 might be true when $\mathcal{P}$ is simple.

References

[1] D. Eisenbud and B. Sturmfels, Binomial ideals, *Duke Math. J.* **84** (1996), 1–46.

[2] J. Herzog and T. Hibi, "Monomial Ideals," GTM 260, Springer, 2011.

[3] J. Herzog, A. A. Qureshi and A. Shikama, Gröbner basis of balanced polyominoes, *Math. Nachr.*, to appear.

[4] J. Herzog and S. S. Madani, The coordinate ring of a simple polyomino, arXiv:1408.4275v1.

[5] T. Hibi, Ed., "Gröbner Bases: Statistics and Software Systems," Springer, 2013.

[6] H. Ohsugi and T. Hibi, Toric ideals generated by quadratic binomials, *J. Algebra* **218** (1999), 509–527.

[7] H. Ohsugi and T. Hibi, Koszul bipartite graphs, *Adv. Appl. Math.* **22** (1999), 25–28.

[8] A. A. Qureshi, H. Shibuta and A. Shikama, Simple polyominoes are prime, arXiv:1501.05107v2.

[9] A. A. Qureshi, Ideals generated by 2-minors, collections of cells and stack polyominoes, *J. Algebra* **357** (2012), 279–303.

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